Paraproducts in One and Several Variables
M. Lacey and J. Metcalfe

Kelly Bickel
Washington University
St. Louis, Missouri 63130

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What is a Paraproduct?!

A **Paraproduct** is a bilinear operator that is similar to, but "nicer" than, a product of two functions. Consider the following operator:

$$\Pi(f,g)(s) := \int_{-\infty}^{s} f'(t)g(t)\, dt, \qquad \forall\, f,g \in C_0^1(\mathbb{R}).$$

Then Π satisfies:

- **Product Reconstruction:** By Leibniz's rule,

$$fg = \Pi(f,g) + \Pi(g,f).$$

- **Linearization Formula:** For $G \in C^\infty(\mathbb{R})$, we have

$$G(f) = G(0) + \Pi(f, G'(f)).$$

- **A Leibniz-type Rule:** It follows immediately that

$$\Pi(f,g)' = f'g.$$

Then Π is *almost* a paraproduct. (generally, paraproducts also satisfy a Hölder's inequality.)

A Paraproduct is . . .

A **Paraproduct** Π is a bilinear operator satisfying: product reconstruction, a linearization formula, a Hölder-type inequality, and a Leibniz-type rule:

$$\partial^\alpha \Pi(f,g) = \Pi'(\partial^\alpha f, g).$$

We study **One-Parameter Model Paraproducts**:

Let I be an interval. Then ϕ_I is a *bump function adapted to* I iff $\|\phi_I\|_2 = 1$ and

$$|D^n \phi_I(x)| \lesssim |I|^{-n-1/2} \left(1 + \frac{|x - c(I)|}{|I|}\right)^{-N}, \qquad n = 0, 1,$$

where $c(I)$ is the center of I and N is sufficiently large. Let $\mathscr{D}$ be the set of dyadic intervals. Define:

$$B(f_1, f_2) := \sum_{I \in \mathscr{D}} |I|^{-1/2} \phi_{3,I} \prod_{j=1}^{2} \langle f_j, \phi_{j,I} \rangle,$$

where, for each I, each $\phi_{j,I}$ is adapted to I and two of the $\phi_{j,I}$ have integral zero.

An Example

For each dyadic interval I, the Haar function adapted to I is

$$h_I := |I|^{-1/2}(\mathbf{1}_{I_l} - \mathbf{1}_{I_r}),$$

where I_l is the left half of I, and I_r is the right half of I. Moreover, define

$$h_I^0 := h_I \quad \text{and } h_I^1 := |h_I|.$$

Then, the **Haar Paraproducts** are given by:

$$B^{k_1,k_2,k_3}(f_1, f_2) = \sum_{I \in \mathscr{D}} |I|^{-1/2} \langle f_1, h_I^{k_1} \rangle \ \langle f_2, h_I^{k_2} \rangle \ h_I^{k_3},$$

where $k_j \in \{0, 1\}$ and two of the k_j are zero. It can be shown that

$$f_1 f_2 = B^{1,0,0}(f_1, f_2) + B^{0,1,0}(f_1, f_2) + B^{0,0,1}(f_1, f_2).$$

Increase the Parameters!

We also study **Bi-parameter Model Paraproducts**:

Let $\mathscr{R}$ be the set of dyadic rectangles in $\mathbb{R}^2$. A function ϕ_R is *adapted to the rectangle R,* where $R = R_1 \times R_2$, if

$$\phi_R(x) = \phi_{R_1}(x_1)\phi_{R_2}(x_2),$$

where each ϕ_{R_k} is adapted to R_k. The bi-parameter model paraproducts are of the form:

$$B(f_1, f_2) = \sum_{R \in \mathscr{R}} |R|^{-1/2} \phi_{3,R} \prod_{j=1}^{2} \langle f_j, \phi_{j,R} \rangle,$$

where each $\phi_{j,R}$ is adapted to R and for each coordinate x_k, $k = 1, 2$, there are two positions in $j = 1, 2, 3$ such that

$$\int_{\mathbb{R}} \phi_{j,R}(x_1, x_2) dx_k = 0 \quad \forall \ x_i \neq x_k \text{ and } \forall \ R \in \mathscr{R}.$$

Then we say B has x_k zeros in the j^{th} position (or $\{\phi_{j,R}\}$ has x_k zeros).

Main Results

The main results proved in the paper are that both the one and bi-parameter model paraproducts satisfy a Hölder-type inequality:

Theorem 1 (Coifman, Meyer '78), (Muscalu, Pipher, Tao, Thiele '04)

Whenever $1 < p_1, p_2 \leq \infty$, $1/r = 1/p_1 + 1/p_2$, and $0 < r < \infty$,

$$\|B(f_1, f_2)\|_r \lesssim \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

Classical Coifman-Meyer Theorem

Theorem 1 is a discrete version of the following one and bi-parameter results:

Let m be a bounded function on $\mathbb{R}^2$, smooth away from the origin and satisfying

$$|\partial^\alpha m(\zeta)| \lesssim \frac{1}{|\zeta|^{|\alpha|}},$$

for sufficiently many multi-indices α and define the bilinear operator $T_m^{(1)}$ by

$$T_m^{(1)}(f,g) = \int_{\mathbb{R}^2} m(\zeta)\hat{f}(\zeta_1)\hat{g}(\zeta_2)e^{2\pi i x(\zeta_1+\zeta_2)}d\zeta,$$

for Schwartz functions $f, g \in \mathcal{S}(\mathbb{R})$. We can generalize this by allowing m to be defined on $\mathbb{R}^{2n}$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 2 (Coifman, Meyer, '78)

If m is a symbol satisfying the above estimates, then the bilinear operator $T_m^{(1)}$ maps $L^p \times L^q \to L^r$ whenever $1 < p, q \leq \infty$, $1/r = 1/p + 1/q$, and $0 < r < \infty$.

Bi-parameter Coifman-Meyer Theorem

Let $m(\zeta, \eta)$ be a bounded function on $\mathbb{R}^4$, smooth away from $\{(\zeta_1, \eta_1) = 0\}$ $\cup\{(\zeta_2, \eta_2) = 0\}$ and satisfying the estimate

$$|\partial_\zeta^\alpha \partial_\eta^\beta m(\zeta, \eta)| \lesssim \frac{1}{|(\zeta_1, \eta_1)|^{\alpha_1+\beta_1}} \frac{1}{|(\zeta_2, \eta_2)|^{\alpha_2+\beta_2}},$$

for sufficiently many multi-indices α and β. Then we can define the bilinear operator $T_m^{(2)}$ as follows:

$$T_m^{(2)}(f, g) = \int_{\mathbb{R}^4} m(\zeta, \eta) \hat{f}(\zeta) \hat{g}(\eta) e^{2\pi i x(\zeta+\eta)} d\zeta d\eta,$$

where $f, g \in \mathcal{S}(\mathbb{R}^2)$.

Theorem 3 (Muscalu, Pipher, Tao, Thiele '04)

If m is a symbol satisfying the above estimates, then the bilinear operator $T_m^{(2)}$ maps $L^p \times L^q \to L^r$ whenever $1 < p, q \leq \infty$, $1/r = 1/p + 1/q$, and $0 < r < \infty$.

Fractional Derivative Estimates

Let $f, g \in \mathcal{S}(\mathbb{R}^2)$ and for $\alpha > 0$, define the fractional derivative $\mathcal{D}^\alpha$ by

$$\widehat{\mathcal{D}^\alpha f}(\zeta) = |\zeta|^\alpha \hat{f}(\zeta).$$

There are paraproducts Π_j for $j = 0, 1, 2, 3$ such that the Coifman-Meyer theorem applies to each Π_j and

$$fg = \sum_{j=0}^{3} \Pi_j(f, g).$$

Using the structure of the Π_j, one can find paraproducts Π_1' and Π_2' with

$$\mathcal{D}^\alpha\big(\Pi_1(f, g)\big) = \Pi_1'(f, \mathcal{D}^\alpha g) \quad \text{and} \quad \mathcal{D}^\alpha\big(\Pi_2(f, g)\big) = \Pi_2'(\mathcal{D}^\alpha f, g),$$

and similar Π_0' and Π_3' paraproducts. Using the Coifman-Meyer theorem, calculate

$$\begin{aligned} ||\mathcal{D}^\alpha(fg)\|_r &\lesssim \sum_{j=0}^{3} \|\mathcal{D}^\alpha(\Pi_j(f, g))\|_r \\ &\lesssim \|\mathcal{D}^\alpha f\|_p \|g\|_q + \|f\|_p \|\mathcal{D}^\alpha g\|_q, \end{aligned}$$

for $1 < p, q \leq \infty$, $1/r = 1/p + 1/q$, and $0 < r < \infty$.

Fractional Partial Derivative Estimates

For $f \in \mathcal{S}(\mathbb{R}^2)$ and for $\alpha, \beta > 0$, define the partial differential operator $\mathcal{D}_1^\alpha \mathcal{D}_2^\beta$ by

$$\widehat{\mathcal{D}_1^\alpha \mathcal{D}_2^\beta f}(\zeta) = |\zeta_1|^\alpha |\zeta_2|^\beta \hat{f}(\zeta).$$

Then, for $f, g \in \mathcal{S}(\mathbb{R}^2)$, using the bi-parameter Coifman-Meyer theorem and analogous manipulations of paraproducts, we have

$$\|\mathcal{D}_1^\alpha \mathcal{D}_2^\beta (fg)\|_r \lesssim \|\mathcal{D}_1^\alpha \mathcal{D}_2^\beta f\|_p \|g\|_q + \|f\|_p \|\mathcal{D}_1^\alpha \mathcal{D}_2^\beta g\|_q$$

$$+ \|\mathcal{D}_1^\alpha f\|_p \|\mathcal{D}_2^\beta g\|_q + \|\mathcal{D}_2^\beta f\|_p \|\mathcal{D}_1^\alpha g\|_q,$$

for $1 < p, q \leq \infty$, $1/r = 1/p + 1/q$, and $0 < r < \infty$.

Square and Maximal Functions

We are considering the following paraproducts:

$$B(f_1, f_2) := \sum_{I \in \mathscr{D}} |I|^{-1/2} \phi_{3,I} \prod_{j=1}^{2} \langle f_j, \phi_{j,I} \rangle,$$

where, for each I, each $\phi_{j,I}$ is adapted to I and two of the $\phi_{j,I}$ have integral zero. Without loss of generality,we can assume $\phi_{2,I}$ and $\phi_{3,I}$ always have integral zero.

We will need the following variations of the maximal function and square function:

$$M_1 g := \sup_{I \in \mathscr{D}} \mathbf{1}_I \frac{|\langle g, \phi_{1,I} \rangle|}{\sqrt{|I|}}$$

$$S_j g := \left[\sum_{I \in \mathscr{D}} \frac{|\langle g, \phi_{j,I} \rangle|^2}{|I|} \mathbf{1}_I \right]^{1/2}, \qquad \text{for } j = 2, 3.$$

Maximal Function Bounds

$$M_1(f)(x) = \sup_{I \in \mathscr{D}} \mathbf{1}_I \frac{|\langle f, \phi_{1,I} \rangle|}{\sqrt{|I|}} \lesssim Mf(x),$$

where M denotes the typical Hardy-Littlewood maximal function. Fix $I \in \mathscr{D}$, say $|I| = 2^{K+1}$. Translate I so that it is centered at 0 and let y be in I. Then:

$$\begin{aligned}
\frac{|\langle f, \phi_I \rangle|}{\sqrt{|I|}} &\lesssim |I|^{-1} \int_{\mathbb{R}} |f(x)| \left(1 + |x|/|I|\right)^{-2} dx \\
&\lesssim |I|^{-1} \int_I |f| + |I|^{-1} \int_{\mathbb{R}-I} |f(x)|(1 + |x|^2/|I|^2)^{-1} dx \\
&\lesssim Mf(y) + \sum_{j>k} 2^{-K-1} \int_{I_j} |f(x)| dx (1 + 2^{2(j-1)}/2^{2(K+1)})^{-1} \\
&\lesssim Mf(y) + \sum_{j>k} 2^{k-j} \frac{1}{2^{j+1}} \int_{I_j} |f(x)| dx \\
&\lesssim Mf(y),
\end{aligned}$$

where I_j is the interval centered around zero with length 2^{j+1}. As $y \in I$, it is clear that $y \in I_j$.

Square Function Bounds

Sf is more or less large only where f is large, which is reflected by

$$\|Sf\|_p \lesssim \|f\|_p \qquad \forall\ 1 < p < \infty.$$

Partially follows because $\langle \phi_I, \phi_{I'} \rangle$ is usually small. In particular, if all $\{\phi_I\}$ have integral zero, then

$$\sum_{I \in \mathscr{D}} |\langle \phi_I, \phi_{I'} \rangle| \leq C.$$

For $p = 2$, restrict to finite sums, let $\|f\|_2 = 1$ and calculate:

$$\sum_I |\langle f, \phi_I \rangle|^2 \leq \| \sum_I \langle f, \phi_I \rangle \phi_I \|_2,$$

using Cauchy-Schwarz and then calculate

$$\begin{aligned} \left(\sum_I |\langle f, \phi_I \rangle|^2 \right)^2 &\leq \sum_I \sum_{I'} \langle f, \phi_I \rangle \langle \phi_{I'}, f \rangle \langle \phi_{I'}, \phi_I \rangle \\ &\leq 2 \sum_I |\langle f, \phi_I \rangle|^2 \sum_{I'} |\langle \phi_{I'}, \phi_I \rangle|. \end{aligned}$$

Proof of One-Parameter Result

Recall: We are trying to show that B maps $L^{p_1} \times L^{p_2} \to L^r$ whenever $1 < p_1, p_2 \leq \infty$, $1/r = 1/p_1 + 1/p_2$, and $0 < r < \infty$.

Case 1: $1 < r < \infty$
In this case, L.M. use a duality argument. We will need the following obvious fact: If $\{a_{j,I}\}_{I \in \mathscr{D}}$ are sequences such that $\{a_{1,I}\} \in l^\infty$, $\{a_{2,I}\}$, $\{a_{3,I}\} \in l^2$, then

$$\sum_{I \in \mathscr{D}} a_{1,I} a_{2,I} a_{3,I} \leq \|a_{1,I}\|_\infty \|a_{2,I}\|_2 \|a_{3,I}\|_2,$$

and in particular,

$$\begin{aligned} \sum_{I \in \mathscr{D}} \prod_{j=1}^{3} \left(\frac{|\langle f_j, \phi_{j,I} \rangle|}{\sqrt{|I|}} \mathbf{1}_I(x) \right) &\leq \sup_{I \in \mathscr{D}} \frac{|\langle f_1, \phi_{1,I} \rangle|}{\sqrt{|I|}} \mathbf{1}_I(x) \prod_{j=2}^{3} \left[\sum_{I \in \mathscr{D}} \frac{|\langle f_j, \phi_{j,I} \rangle|^2}{|I|} \mathbf{1}_I(x) \right]^{1/2} \\ &= (M_1 f_1)(S_2 f_2)(S_3 f_3)(x) \end{aligned}$$

Proof of One-Parameter Result

Case 1: $1 < r < \infty$

Let r' be dual to r, fix $f_3 \in L^{r'}$ with $\|f_3\|_{r'} = 1$, and calculate:

$$\begin{aligned}
\langle B(f_1, f_2), f_3 \rangle &= \int \sum_{I \in \mathscr{D}} |I|^{-1/2} \phi_{3,I} \prod_{j=1}^{2} \langle f_j, \phi_{j,I} \rangle \bar{f}_3 \\
&\leq \int \sum_{I \in \mathscr{D}} |I|^{-3/2} \prod_{j=1}^{3} |\langle f_j, \phi_{j,I} \rangle| \mathbf{1}_I \\
&\lesssim \int (Mf_1)(S_2 f_2)(S_3 f_3) \\
&\leq \|Mf_1\|_{p_1} \|S_2 f_2\|_{p_2} \|S_3 f_3\|_{r'} \\
&\lesssim \|f_1\|_{p_1} \|f_2\|_{p_2},
\end{aligned}$$

where Hölder's inequality is used. Taking the supremum over all such f_3 yields:

$$\|B(f_1, f_2)\|_r \lesssim \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

Proof of One-Parameter Result

Case 2: $1/2 < r < 1$

M.L. prove the following weak-type estimates:

$$\lambda |\{B(f_1, f_2) > \lambda\}|^{1/r} \lesssim \|f_1\|_{p_1} \|f_2\|_{p_2}, \tag{1}$$

and multi-linear Marcinkiewicz interpolation yields the desired strong estimates. To get the weak estimates, use the following lemma:

Lemma 1 (Auscher, Hofmann, Muscalu, Tao, Thiele '02)

Let $0 < r < \infty$. If for all sets E with $0 < |E| < \infty$, there is subset $E' \subseteq E$ with $|E'| \sim |E|$ and $|\langle f, \mathbf{1}_{E'} \rangle| \lesssim A|E|^{1/r'}$ then

$$\|f\|_{r,\infty} \lesssim A.$$

In particular, let $f = B(f_1, f_2)$ and $A = \|f_1\|_{p_1} \|f_2\|_{p_2}$.

The Λ operator

Consider the following multi-linear operator:

$$\Lambda(f_1, f_2, f_3) := \sum_{I\in\mathscr{D}} |I|^{-1/2} \prod_{j=1}^{3} |\langle f_j, \phi_{j,I}\rangle|. \tag{2}$$

Then, M.L. shows that for each f_1, f_2, and set E, there is a set $E' \subseteq E$ with $|E'| \sim |E|$ such that

$$\Lambda(f_1, f_2, f_3) \lesssim |E|^{1/r'} \|f_1\|_{p_1} \|f_2\|_{p_2},$$

for all f_3 supported in E' and bounded by 1. (particularly, $f_3 = \mathbf{1}_{E'}$.)

By multi-linearity, we can assume $\|f_1\|_{p_1} = \|f_2\|_{p_2} = 1$. As the class of the multi-linear forms Λ is invariant under dilations by powers of two, we can assume $|E| = 1$.

Specifically, if D_λ is the dilation operator defined by $(D_\lambda f)(x) = f(\lambda^{-1}x)$ and

$$\Lambda^k(f_1, f_2, f_3) := 2^{-k}\Lambda(D_{2^k} f_1, D_{2^k} f_2, D_{2^k} f_3),$$

then Λ^k is a multi-linear form of type (2) for $k \in \mathbb{Z}$.

End!

The End!

Paraproducts in One and Several Variables (Part II)
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Bi-parameter Paraproducts- Review

Recall: for I be an interval, we say ϕ_I is a *bump function adapted to I* iff $\|\phi_I\|_2 = 1$ and

$$|D^n \phi_I(x)| \lesssim |I|^{-n-1/2} \left(1 + \frac{|x - c(I)|}{|I|}\right)^{-N}, \qquad n = 0, 1,$$

where $c(I)$ is the center of I and N is sufficiently large. Last time, we considered paraproducts of the form:

$$B(f_1, f_2) := \sum_{I \in \mathscr{D}} |I|^{-1/2} \phi_{3,I} \prod_{j=1}^{2} \langle f_j, \phi_{j,I} \rangle,$$

where, for each I, each $\phi_{j,I}$ is adapted to I and two of the $\phi_{j,I}$ have integral zero.

Let $\mathscr{R}$ be the set of dyadic rectangles in $\mathbb{R}^2$. A function ϕ_R is *adapted to the rectangle R,* where $R = R_1 \times R_2$, if

$$\phi_R(x) = \phi_{R_1}(x_1)\phi_{R_2}(x_2),$$

where each ϕ_{R_k} is adapted to R_k.

Bi-parameter Paraproducts- Review

The bi-parameter model paraproducts are of the form:

$$B(f_1, f_2) = \sum_{R \in \mathscr{R}} |R|^{-1/2} \phi_{3,R} \prod_{j=1}^{2} \langle f_j, \phi_{j,R} \rangle,$$

where each $\phi_{j,R}$ is adapted to R and for each coordinate x_k, $k = 1, 2$, there are two positions in $j = 1, 2, 3$ such that

$$\int_{\mathbb{R}} \phi_{j,R}(x_1, x_2) dx_k = 0 \quad \forall\ x_i \neq x_k \text{ and } \forall\ R \in \mathscr{R}.$$

Then we say B has x_k zeros in the j^{th} position (or $\{\phi_{j,R}\}$ has x_k zeros).

Theorem 2 (Muscalu, Pipher, Tao, Thiele '04)

Whenever $1 < p_1, p_2 \leq \infty$, $1/r = 1/p_1 + 1/p_2$, and $0 < r < \infty$,

$$\|B(f_1, f_2)\|_r \lesssim \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

Variants of Square and Maximal functions

Again, M.L. use variants of square and maximal functions, adapted to the specific bump functions appearing in the given paraproduct B.

Consider the following iterates of one-variable square and maximal functions:

$$MM(f) := \sup_{R \in \mathscr{R}} \frac{|\langle f, \phi_R \rangle|}{\sqrt{|R|}} \mathbf{1}_R$$

$$S_1 M_2(f) := \left[\sum_{R_1 \in \mathscr{D}} \sup_{R_2 \in \mathscr{D}} \frac{|\langle f, \phi_{R_1 \times R_2} \rangle|^2}{|R|} \mathbf{1}_R \right]^{1/2}, \qquad R = R_1 \times R_2$$

$$SS(f) := \left[\sum_{R \in \mathscr{R}} \frac{|\langle f, \phi_R \rangle|^2}{|R|} \mathbf{1}_R \right]^{1/2},$$

where we can similarly define $S_2 M_1$, $M_1 S_2$, and $M_2 S_1$.

If a square function is applied to the set $\{\phi_R\}$ in the x_k coordinate, we require the functions $\{\phi_R\}$ to have x_k zeros.

Biparameter Proof: Case 1

As before, the interated square and maximal functions are bounded from $L^p \to L^p$, for $1 < p < \infty$. Specficically, if T is an operator on the previous slide,

$$\|Tf\|_p \quad \lesssim \quad \|f\|_p$$

for $1 < p < \infty$. As before, we define the multilinear form Λ by

$$\Lambda(f_1, f_2, f_3) = \sum_{R \in \mathscr{R}} |R|^{-1/2} \prod_{j=1}^{3} |\langle f_j, \phi_{j,R} \rangle|$$

for $f_3 \in L^{r'}$, and have:

$$\langle B(f_1, f_2), f_3 \rangle \quad \leq \quad \Lambda(f_1, f_2, f_3)$$

$$= \quad \int \sum_{R \in \mathscr{R}} \prod_{j=1}^{3} |R|^{-1/2} |\langle f_j, \phi_{j,R} \rangle| \mathbf{1}_R,$$

We will use operators to bound the sum inside the integral. The operators we choose will depend on where B has zeros in each coordinate.

To illustrate, assume B has x_2 zeros in the $j = 1, 2$ positions and x_1 zeros in the $j = 2, 3$ positions. Then we have:

$$\sum_{R \in \mathscr{R}} \prod_{j=1}^{3} |R|^{-1/2} |\langle f_j, \phi_{j,R} \rangle| \mathbf{1}_R$$

$$\leq \sum_{R_1 \in \mathscr{D}} \sup_{R_2 \in \mathscr{D}} \frac{|\langle f_3, \phi_{3,R} \rangle|}{\sqrt{|R|}} \mathbf{1}_R \prod_{j=1}^{2} \left(\sum_{R_2 \in \mathscr{D}} \frac{|\langle f_j, \phi_{j,R} \rangle|^2}{|R|} \mathbf{1}_R \right)^{1/2}$$

$$\leq \sup_{R_1} \left(\sum_{R_2} \frac{|\langle f_1, \phi_{1,R} \rangle|^2}{|R|} \mathbf{1}_R \right)^{\frac{1}{2}} \left(\sum_{R} \frac{|\langle f_2, \phi_{2,R} \rangle|^2}{|R|} \mathbf{1}_R \right)^{\frac{1}{2}} \left(\sum_{R_1} \sup_{R_2} \frac{|\langle f_3, \phi_{3,R} \rangle|^2}{|R|} \mathbf{1}_R \right)^{\frac{1}{2}}$$

$$= (M_1 S_2 f_1)(SSf_2)(S_1 M_2 f_3)$$

$$\leq (S_2 M_1 f_1)(SSf_2)(S_1 M_2 f_3).$$

Bi-parameter Proof Case 1

In general, there are 3 iterated square/maximal operators T_j for $j = 1, 2, 3$ with

$$\begin{aligned}
\langle B(f_1, f_2), f_3\rangle &\leq \Lambda(f_1, f_2, f_3) \\
&= \int \sum_{R\in\mathscr{R}} \prod_{j=1}^{3} |R|^{-1/2} |\langle f_j, \phi_{j,R}\rangle| \mathbf{1}_R \\
&\leq \int T_1 f_1 \; T_2 f_2 \; T_3 f_3.
\end{aligned}$$

Case 1: For $1 < r < \infty$, let r' be dual to r and choose $f_3 \in L^{r'}$ with $\|f_3\|_{r'} = 1$. Then

$$\begin{aligned}
\langle B(f_1, f_2), f_3\rangle &\leq \|T_1 f_1\|_{p_1} \|T_2 f_2\|_{p_2} \|T_3 f_3\|_{r'} \\
&\lesssim \|f_1\|_{p_1} \|f_2\|_{p_2},
\end{aligned}$$

which gives the result for $1 < r < \infty$.

Bi-parameter Proof Case 2

Case 2: $1/2 < r < 1$

M.L. prove the following weak-type estimates:

$$\lambda |\{B(f_1, f_2) > \lambda\}|^{1/r} \lesssim \|f_1\|_{p_1} \|f_2\|_{p_2},$$

and multi-linear Marcinkiewicz interpolation yields the desired strong estimates.

To get the weak estimates, show, that for each f_1, f_2 with $\|f_1\|_{p_1} = \|f_2\|_{p_2} = 1$ and set E with $|E| = 1$, there is a set E' with $E' \subseteq E$ with $|E'| \sim |E|$ and

$$\Lambda(f_1, f_2, f_3) \lesssim 1,$$

for every f_3 supported in E' and bounded by 1.

Further, we can assume each f_j is smooth and compactly supported. Let T_j, for $j = 1, 2, 3$, be the operators bounding B as in the previous slide.

$\mathscr{O}$ Notation

We will be estimating

$$\Lambda(f_1, f_2, f_3) = \sum_{R \in \mathscr{R}} |R|^{-1/2} \prod_{j=1}^{3} |\langle f_j, \phi_{j,R} \rangle|.$$

In particular, we will decompose $\mathscr{R}$ into several classes of rectangles. Let $\mathscr{O}$ be a class of dyadic rectangles. Then define

$$\text{sum}(\mathscr{O}) = \sum_{R \in \mathscr{O}} |R|^{-1/2} \prod_{j=1}^{3} |\langle f_j, \phi_{j,R} \rangle|.$$

Recall that for each iterated operator T_j we were summing (or sup-ing) over $\langle f, \phi_R \rangle$, for $R \in \mathscr{R}$. Let $T_{\mathscr{O}}$ denote an iterated square or maximal function restricted to the class of dyadic rectangles $\mathscr{O}$. For example, if $T = SS$,

$$T_{\mathscr{O}} f = \left(\sum_{R \in \mathscr{O}} \frac{|\langle f, \phi_R \rangle|^2}{|R|} \mathbf{1}_R \right)^{1/2}.$$

Before we define E', we need to establish several bounds on $\text{sum}(\mathscr{O})$ and $\| T_{\mathscr{O}} \|_2$ for classes of rectangles $\mathscr{O}$ satisfying special properties.

Technical Lemma 1

Lemma 2

Let $\mathscr{O} \subseteq \mathscr{R}$ and let $\mu > 1$ be a constant such that $\text{supp}(f) \cap \mu R = \emptyset \ \forall \ R \in \mathscr{O}$, for a given function f. Then

$$\|T_{\mathscr{O}} f\|_2 \lesssim \mu^{-N'} \|f\|_2,$$

where $N' = N - N_0$, where N is the integer in the definition of adapted for the $\{\phi_R\}$ defining T, and N_0 is the smallest integer needed to get the L^p bounds on the square and maximal functions.

Idea of Proof Let $\{\phi_R\}$ be *adapted* with integer $N > 0$. One can define a new set of adapted functions $\{\tilde{\phi}_R\}$ adapted with integer N_0 such that

$$\tilde{\phi}_R(x) = \mu^{N'} \phi_R(x) \qquad \forall \ x \notin \mu R.$$

Define T with the $\{\phi_R\}$ and $\tilde{T}$ with the $\{\tilde{\phi}_R\}$. If f satisfies the assumptions of the lemma, then

$$T_{\mathscr{O}} f = \mu^{-N'} \tilde{T}_{\mathscr{O}} f,$$

and the result follows since $\tilde{T}$ is bounded on L^2, with bounds independent of μ.

Technical Lemma 2

Lemma 3

Let $c_1, c_2, c_3 > 0$ be constants and $\mathscr{O}$ a collection of rectangles such that

$$|R \cap \{T_j f_j > c_j\}| \leq \tfrac{1}{100}|R| \qquad \text{for } R \in \mathscr{O}, j = 1, 2, 3. \tag{3}$$

Then we have the estimate: $\quad \operatorname{sum}(\mathscr{O}) \lesssim c_1 c_2 c_3 |\operatorname{sh}\mathscr{O}|.$

If (3) is not known for $j = 3$, we have: $\quad \operatorname{sum}(\mathscr{O}) \lesssim c_1 c_2 |\operatorname{sh}\mathscr{O}|^{1/2} \| T_{3\mathscr{O}} f_3 \|_2.$

Idea of Proof:

Let $W = \operatorname{sh}(\mathscr{O}) \cap \bigcap_{j=1}^{3} \{T_j f_j < c_j\}$, so that $|R \cap W| \geq \frac{97}{100}|R|$. Then:

$$\begin{aligned}
\operatorname{sum}(\mathscr{O}) &\lesssim \int_W \sum_{R \in \mathscr{O}} \prod_{j=1}^{3} |R|^{-1/2} |\langle f_j, \phi_{j,R} \rangle| \mathbf{1}_R \\
&\leq \int_W T_1 f_1 \; T_2 f_2 \; T_3 f_3 \\
&\leq |\operatorname{sh}(\mathscr{O})| c_1 c_2 c_3.
\end{aligned}$$

Definition of E'

Fix $f_1,\ f_2,\ E$ with $\|f_1\|_{p_1} = \|f_2\|_{p_2} = |E| = 1$. Define $4\nu = \min(p_1, p_2)$ and let T_0 be the strong maximal function (in two parameters). Define

$$\begin{aligned}
\Omega_{j,l} &:= \{T_j f_j > C2^l\}, \qquad l \in \mathbb{Z},\ j = 1, 2, \\
\Omega_l &:= \cup_{j=1}^{2} \Omega_{j,l}, \\
\Omega &:= \cup_{l\in\mathbb{N}} \{T_0 \mathbf{1}_{\Omega_l} > 2^{-\nu l}/100\}, \\
\tilde{\Omega} &:= \{T_0 \mathbf{1}_{\Omega} > 1/2\}.
\end{aligned}$$

Set $E' = \tilde{\Omega}^c \cap E$. We can choose C so that $|E'| \geq 1/2$ by choosing C so that $|\Omega| < 1/8$. Using the L^2 boundedness of T_0 and L^{p_j} boundedness of the T_j for $j = 1, 2$, we have

$$|\Omega| \leq K_1 \sum_{l\in\mathbb{N}} |\Omega_l| 2^{2\nu l} \leq K_2 \sum_{l\in\mathbb{N}} \sum_{j=1}^{2} C^{-p_j} 2^{l(2\nu - p_j)},$$

which converges, and so we can choose $C >> 0$ to give $\tilde{\Omega}$ the desired size.

Decomposition of $\mathscr{R}$

Recall, we are trying to show:

$$\Lambda(f_1, f_2, f_3) = \sum_{R \in \mathscr{R}} |R|^{-1/2} \prod_{j=1}^{3} |\langle f_j, \phi_{j,R} \rangle| \lesssim 1,$$

where f_3 is bounded by one and supported on E'. Then, for $1 < p_3 < \infty$, $\|f_3\|_{p_3} \leq 1$.

We consider the sum restricted to specific classes of rectangles in $\mathscr{R}$ and split the rectangles into classes as follows:

R is in class $\mathscr{O}_{j,l}$ iff l is the greatest integer so that

$$|R \cap \Omega_{j,l}| = |R \cap \{T_j f_j > C2^l\}| \geq \frac{1}{100}|R|.$$

As the $T_j f_j$ are bounded, every rectangle R is in precisely one $\mathscr{O}_{j,l}$ for each j and so we can associate to each R a tuple $\vec{l} = (l_1, l_2, l_3)$ of integers.

$\vec{l}$ with $l_1, l_2 \leq 0$

Let L denote the tuples with $l_1, l_2, l_3 \leq 0$. Fix such an $\vec{l} = (l_1, l_2, l_3)$ and define

$$\mathscr{O}_l = \cap_{j=1}^{3} \mathscr{O}_{j,l_j}.$$

Then for each $R \in \mathscr{O}_l$, and $j = 1, 2, 3$,

$$|R \cap \Omega_{j,l_j+1}| = |R \cap \{T_j f_j > C2^{l_j+1}\}| < \frac{1}{100}|R|.$$

and so Technical Lemma 2 yields:

$$\text{sum}(\mathscr{O}_l) \lesssim |\text{sh}(\mathscr{O}_l)| 2^{l_1+l_2+l_3}.$$

The L^{p_j}-boundedness of T_j implies that, for $\theta_1 + \theta_2 + \theta_3 = 1$,

$$\begin{aligned} |\text{sh}(\mathscr{O}_l)| &\leq |\text{sh}(\mathscr{O}_{1,l_1})|^{\theta_1} |\text{sh}(\mathscr{O}_{2,l_2})|^{\theta_2} |\text{sh}(\mathscr{O}_{3,l_3})|^{\theta_3} \\ &\lesssim 2^{-p_1 l_1 \theta_1 - p_2 l_2 \theta_2 - p_3 l_3 \theta_3}. \end{aligned}$$

Then we can calculate

$$\sum_{l \in L} \text{sum}(\mathscr{O}_l) \lesssim \sum_{l \in L} 2^{l_1(1-p_1\theta_1)+l_2(1-p_2\theta_2)+l_3(1-p_3\theta_3)},$$

which converges for $\theta_1, \theta_2, \theta_3$ and $p_3 > 0$ with $1 - p_j\theta_j > 0$.

$\vec{l}$ with $l_1 > 0$ or $l_2 > 0$

Let G denote the tuples with at least one of $l_1, l_2 \geq 0$. Fix such an $\vec{l} = (l_1, l_2, l_3)$ and define

$$\mathscr{O}_l = \cap_{j=1}^{2} \mathscr{O}_{j,l_j}.$$

Fix such an l and without loss of generality, assume $l_1 > 0$. Let $R \in \mathscr{O}_l$. Let $2^{\nu l_1/2} R$ be the rectangle obtained by dilating each side of R by a factor of $2^{\nu l_1/2}$ and keeping the same center. Then

$$\begin{aligned} \frac{1}{|2^{\nu l_1/2} R|} \int_{2^{\nu l_1/2} R} \mathbf{1}_{\Omega_{l_1}} &= |2^{\nu l_1/2} R \cap \Omega_{l_1}| / |2^{\nu l_1/2} R| \\ &\geq |R \cap \Omega_{l_1}| / 2^{\nu l_1} |R| \\ &\geq 2^{-\nu l_1}/100, \end{aligned}$$

which implies $2^{\mu l_1/2} R \subseteq \Omega$ and so

$$2^{\nu l_1/2} R \cap \operatorname{supp}(f_3) = \emptyset \qquad \forall\ R \in \mathscr{O}_l.$$

Technical Lemma 1 gives:

$$\| T_{\mathscr{O}_l,3} f_3 \|_2 \lesssim 2^{-N' \nu l_1/2} \| f_3 \|_2 \leq 2^{-10 l_1},$$

for N' sufficiently large.

$\vec{l}$ with $l_1 > 0$ or $l_2 > 0$

Actually, we showed:

$$\| T_{\mathscr{O}_l,3} f_3 \|_2 \lesssim \min(2^{-10 l_1}, 2^{-10 l_2}).$$

Now, as each $R \in \mathscr{O}_l$ satisfies:

$$|R \cap \{ T_j f_j > C 2^{l_j + 1} \}| \leq \tfrac{1}{100} |R| \qquad \text{for } R \in \mathscr{O}_l, \ j = 1, 2,$$

Technical Lemma 2 implies:

$$\begin{aligned} \operatorname{sum}(\mathscr{O}_l) &\lesssim 2^{l_1 + l_2} |\operatorname{sh} \mathscr{O}_l|^{1/2} \| T_{\mathscr{O}_l,3} f_3 \|_2 \\ &\lesssim 2^{l_1 + l_2} \min(2^{-10 l_1}, 2^{-10 l_2}), \end{aligned}$$

which is clearly summable over all tuples (l_1, l_2) with l_1 or l_2 positive. This covers the entire class of dyadic rectangles. Thus, we have proved:

$$\Lambda(f_1, f_2, f_3) = \sum_{R \in \mathscr{R}} |R|^{-1/2} \prod_{j=1}^{3} |\langle f_j, \phi_{j,R} \rangle| \lesssim 1,$$

as desired.